Foundations of the Lindbladian Approach to Reversible Computing

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Outline

- Nonequilibrium Landauer principle, thermal operations, Kraus operators.
- GKSL (Lindbladian) dynamics with multiple asymptotic states.
- Applications to reversible computing.

Modelling Reversible Computational Processes

- Want to represent computations via operations on quantum states.
  - For S in state $\hat{\rho}_S$ coupled to B at state $\hat{\rho}_{B,\beta}$, thermal operation is a map on $\mathcal{D}(\mathcal{H}_S)$:

$$\mathcal{E}(\hat{\rho}_S): \hat{\rho}_S \mapsto \mathrm{Tr}_B \left\{ \hat{U}_{SB} \left(\hat{\rho}_S \otimes \hat{\rho}_{B,\beta} \right) \hat{U}_{SB}{}^{\dagger} \right\}.$$

    - Require $\left[\hat{U}_{SB}, \hat{H}\right] = \mathbf{0}$ over all times. (Otherwise, $\hat{U}_{SB}$ requires work[2] to perform.)
- Standard[2] setup: N distinguishable density matrices encoding N-ary alphabet.
  - Exploiting convex linear combination property, each density matrix can be expressed in terms of a *decoherence free subspace* (DFS) of a larger dynamical system.

[1] – M. Lostalgio, Á. Alhambra, and C. Perry, Quantum 2, 52 (2018).
[2] – N. Anderson, Eur. Phys. J. B 91, 156 (2018)

Nonequilibrium Landauer Limit

- Explicit nonequilibrium Landauer bound[3,4–5] via thermal operations:

$$\beta\langle Q\rangle \geq -\ln \mathrm{Tr}\left\{(\hat{\rho}_S(0)\otimes \mathbb{1}_B)\,\hat{U}_{SB}{}^{\dagger}\,(\mathbb{1}_S\otimes\hat{\rho}_{B,\beta})\,\hat{U}_{SB}\right\} = -\ln \mathrm{Tr}\left\{\sum\nolimits_{\ell}\hat{E}_\ell\,\hat{E}_\ell{}^{\dagger}\hat{\rho}_{B,\beta}\right\}$$

  - Ejection of information in correlated bits[4,6]: loss of prior correlations to environment.

- Dissipation bound depends on subsystem non-unitality of $\hat{U}_{SB}$.
  - Bath measure of non-unitality[5]: $\mathcal{N}_B \coloneqq \left\|\sum_\ell \hat{E}_\ell\,\hat{E}_\ell{}^{\dagger} - \mathbb{1}_E\right\|_2$
    - Unital quantum channel maps $\mathbb{1}$ to $\mathbb{1}$. *Kraus operators* $\hat{E}_\ell$ satisfy $\sum_\ell \hat{E}_\ell\,\hat{E}_\ell{}^{\dagger} = \mathbb{1}$. Satisfy $\mathcal{N}_B = 0$: no lower bound on dissipation if unital.
  - $\hat{E}_\ell = \langle s_a|\hat{U}_{SB}|s_b\rangle$: (environment) Kraus operators. Maps $\mathcal{D}(\mathcal{H}_B)$ to itself (via operator algebra).

[3] – J. Goold, M. Paternostro, and K. Modi, Phys. Rev. Lett. 114, 060602 (2015).
[4] – P. Faist *et al.*, Nat. Comm 6, 7669 (2015).
[5] – G. Guarnieri *et al.*, New J. Phys. 19, 103038 (2017).
[6] – M. P. Frank, arXiv:1806.10183.

Kraus Operators

- *Kraus (measurement) operators:* let us express evolution of coupled SB purely in terms of operators on one of the subsystems. Evolution:

$$\hat{\rho}_S(t) \mapsto \mathrm{Tr}_B\left\{\hat{U}_{SB}\left(\hat{\rho}_S \otimes \hat{\rho}_B\right)\hat{U}_{SB}{}^{\dagger}\right\} = \sum_{jk} \langle b_k|\hat{U}_{SB}(t)|b_j\rangle\, \hat{\rho}_S(0)\, \left\langle b_j\middle|\hat{U}_{SB}{}^{\dagger}(t)\middle|b_k\right\rangle$$

  - $\hat{E}_{jk} \coloneqq \sqrt{\lambda_j}\,\langle b_k|\hat{U}_{SB}|b_j\rangle$ are (system) Kraus operators over environment eigenstates $\{|b_i\rangle\}$.
  - More general: no restriction on $\left[\hat{U}_{SB}, \hat{H}\right]$ and no requirement that B starts in a thermal state.
  - Any map of the form $\hat{\rho}_S(t) \mapsto \sum_{jk} \hat{E}_{jk}(t)\,\hat{\rho}_S(0)\,\hat{E}_{jk}{}^{\dagger}(t)$ is a *CPTP map.*
  - $\hat{E}_{jk} \in \mathbf{Op}(\mathcal{H}_S)$: takes $\mathcal{D}(\mathcal{H}_S)$ to itself, but expression depends on $\{|b_i\rangle\}$.

Lindbladian (GKSL) Evolution

- Markov approximation for evolution: $\mathcal{E}_{\mathrm{d}t}[\hat{\rho}_S(t)] \coloneqq \hat{\rho}_S(t+\mathrm{d}t) = \sum_\ell \hat{E}_\ell(\mathrm{d}t)\,\hat{\rho}(t)\,\hat{E}_{jk}{}^\dagger(\mathrm{d}t)$.

  Expansion of $\mathcal{E}_{\mathrm{d}t}$ in $\mathrm{d}t$: $\mathcal{E}_{\mathrm{d}t} = \mathcal{I} + \mathrm{d}t\,\mathcal{L} + \cdots$, so $\mathcal{L} \coloneqq \lim_{\mathrm{d}t\to 0}(\mathcal{E}_{\mathrm{d}t} - \mathcal{I})/\mathrm{d}t$.

- For $\mathcal{E}_{\mathrm{d}t}$ expansion, need expansion of Kraus operators in $\mathrm{d}t$. Defines *jump operators* $\{\hat{F}_\ell\}$:

$$\hat{E}_0(\mathrm{d}t) = \mathbb{1}_S - i\hat{H}_S\,\mathrm{d}t - \frac{1}{2}\sum_{\ell>0}\mathrm{d}t\,\hat{F}_\ell{}^\dagger\hat{F}_\ell \qquad \hat{E}_{\ell>0} = \sqrt{\kappa_\ell\,\mathrm{d}t}\,\hat{F}_\ell$$

  - $\hat{F}_\ell$ allow us to capture evolution outside of $\hat{H}_S$ and express it in terms of DE for $\hat{\rho}_S$ evolution.

- $\mathcal{L}$ is *Lindbladian / GKSL equation*: $\hat{\rho}_S$ evolution DE. Formal solution: $\hat{\rho}_S(t) = e^{t\mathcal{L}}[\hat{\rho}_S(0)]$

$$\mathrm{d}\hat{\rho}_S/_{\mathrm{d}t} \eqqcolon \mathcal{L}[\hat{\rho}_S(t)] = -i\left[\hat{H}_S, \hat{\rho}_S\right] + \frac{1}{2}\sum_{\ell>0}\kappa_\ell\left(2\hat{F}_\ell\,\hat{\rho}_S\,\hat{F}_\ell{}^\dagger - \left\{\hat{F}_{jk}{}^\dagger\,\hat{F}_{jk}, \hat{\rho}_S\right\}\right)$$

Spectrum of GKSL and Berry Phase

- Evolution: $\hat{\rho}_S(t) = e^{t\mathcal{L}}[\hat{\rho}_S(0)]$. *Asymptotic / steady state*: $\hat{\rho}_{SS} \coloneqq \lim_{t\to\infty} e^{t\mathcal{L}}[\hat{\rho}_S(0)]$.
  - Usually, assume only one steady state, defined as right eigenvector of $\mathcal{L}$ with eigenvalue zero.
- *Multiple asymptotic states*[7–9]: *all* right eigenvectors with pure imaginary eigenvalues give nondecaying $\hat{\rho}_S(t)$. ($\mathfrak{Re} < 0$: decays. $\mathfrak{Re} > 0$: unphysical.)
  - Defines an *asymptotic subspace* $\mathsf{As}(\mathsf{H})$ which can support nontrivial dynamics internally.
- Time evolution via adiabatic approximation. $\hat{H}(\lambda^\mu)$ parametrized by $\lambda^\mu(t) \in \mathbb{R}^n$:

$$|\psi(t)\rangle = e^{i\phi_n(t)} \exp\left\{-i\int_0^t \mathrm{d}\tau\, E_n(\lambda^\mu)\right\} |n(\lambda^\mu)\rangle$$

- Closed path in parameter space $\mathbb{R}^n$: returning to starting conditions. Permits nonzero $U(1)$ gauge-invariant ϕ_n: Berry phase.
  - Degenerate eigenspaces: ϕ_n upgraded to trace of path ordered exponential integral of a $U(N)$ matrix.

[7] – V. V. Albert and L. Jiang, Phys. Rev. A 89, 022118 (2014).
[8] – V. V. Albert *et al.*, Phys. Rev. X 6, 041031 (2016).

Berry Connection and the Quantum Geometric Tensor

- *Berry connection / potential / 1-form*: gauge-dependent connection from Berry phase.

$$\phi_n = \oint_C \mathrm{d}\lambda^\mu \, A_\mu \, ; \quad A_\mu = \langle n(\lambda) | \partial_\mu | n(\lambda) \rangle; \quad |n\rangle \mapsto e^{i\xi} |n\rangle \Leftrightarrow A_\mu \mapsto A_\mu - \partial_\mu \xi$$

- Distance on manifold sketched out by variation of $\widehat{H}$ in parameter space[9]:

$$\mathrm{d}s^2 = \|\psi(\lambda + \mathrm{d}\lambda) - \psi(\lambda)\|^2 = \langle \partial_\mu \psi | \partial_\nu \psi \rangle \, \mathrm{d}\lambda^\mu \, \mathrm{d}\lambda^\nu = \left(\gamma_{\mu\nu} + i\sigma_{\mu\nu}\right) \mathrm{d}\lambda^\mu \, \mathrm{d}\lambda^\nu$$

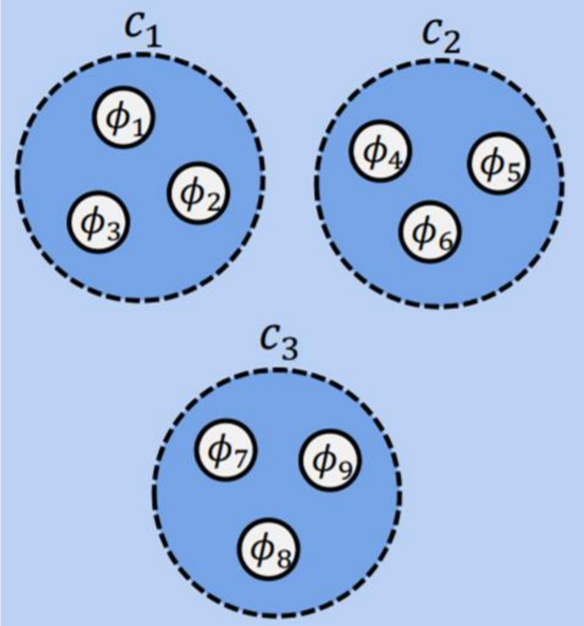
  - $\sigma_{\mu\nu}$: Berry curvature. Curvature of manifold, defined by $\sigma_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.
  - $g_{\mu\nu} = \gamma_{\mu\nu} - A_\mu \wedge A_\nu$: quantum geometric tensor. ($\gamma_{\mu\nu}$ is metric tensor on $\mathcal{H}$, $g_{\mu\nu}$ on $\mathcal{H}/U(1)$).

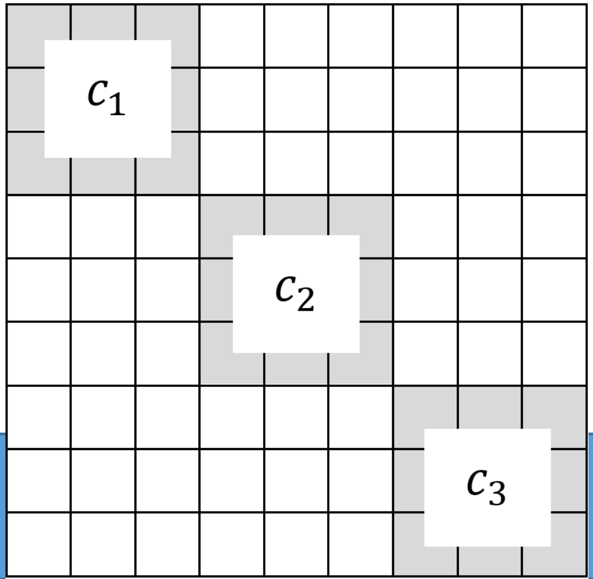
- ϕ_n, A_μ, $\sigma_{\mu\nu}$, and $g_{\mu\nu}$ arose from adiabatic approximation of evolution. GKSL systems with multiple steady states give[8] analogous quantities in the space of operators on $\mathcal{H}$.

[8] – V. V. Albert *et al.*, Phys. Rev. X 6, 041031 (2016).
[9] – R. Cheng, arXiv:1012.1337.

Application to Classical Reversible Computing

- Asymptotic subspace can support direct sums of decoherence free subspaces, each representing a single computational state.
  - Nontrivial asymptotic subspaces support quantum dynamics: quantum computation, but also classical reversible computation *a la* N. Anderson.
  - GKSL evolution describes echoes of initial state upon asymptotic dynamics.
- Nontrivial operator space manifold QGT for all systems beyond single asymptotic state.
  - *Will* show up in dissipation bounds and quantities of interest: e.g. thermodynamic uncertainty relations[10], thermodynamic length[11].





[10] – G. Guarnieri *et al.*, Phys. Rev. Res. 1, 033021 (2019).
[11] – M. Scandi and M. Perarnau-Llobet, Quantum 3, 197 (2019).

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